EPFL– Fall 2025
Series 4

Differential Geometry I: Curves and surfaces

G. Moschidis
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4.1 If a and b are vectors in $\mathbb{R}^3$ with $a \neq 0$, we say that a vector $c \in \mathbb{R}^3$ is the normal component of $b \in \mathbb{R}^3$ relative to $a \in \mathbb{R}^3$ if $c \perp a$ and there exists $\lambda \in \mathbb{R}$ such that $b = \lambda a + c$. Show that this normal component can be written as

$$c = \frac{(a \times b) \times a}{\|a\|^2}.$$

If $\gamma : I \to \mathbb{R}^n$, prove that its curvature (i.e. the norm of the curvature vector) $\gamma : I \to \mathbb{R}^3$:

$$\kappa_\gamma(u) = \frac{\|\dot{\gamma}(u) \times \ddot{\gamma}(u)\|}{V_\gamma^3}.$$

4.2 Let $\alpha : I \to \mathbb{R}^n$ be a biregular curve, with curvature vector K_α and principal normal N_α. Prove that $K_\alpha = \kappa_\alpha N_\alpha$, where κ_α is the curvature of α.

4.3 Prove that the curve $\gamma(t) = (\cosh(t), \sinh(t), t)$ is biregular, then compute its curvature vector and its curvature (the curvature is the norm of the curvature vector).

4.4 The *evolute* of a biregular curve $\alpha : I \to \mathbb{R}^n$ is the curve $\beta : I \to \mathbb{R}^n$ defined by

$$\beta(u) = \alpha(u) + \frac{1}{\kappa_\alpha(u)} N_{\alpha,u},$$

where $\rho_\alpha(u) = \frac{1}{\kappa_\alpha(u)}$ is the radius of curvature and $N_{\alpha,u}$ is the principal normal vector. The evolute of a curve is thus the locus of its centers of curvature (= centers of osculating circles).

Compute the evolutes of the following curves:

(a) A circle in $\mathbb{R}^n$.
(b) A straight line in $\mathbb{R}^n$.
(c) The circular helix $\alpha(u) = (a\cos(u), a\sin(u), bu)$ in $\mathbb{R}^3$.
(d) The cycloid $\gamma(t) = (r(t - \sin t), r(1 - \cos t))$ in $\mathbb{R}^2$.

Prove that the evolute of the helix is again a helix, and that the evolute of the cycloid is again a cycloid.

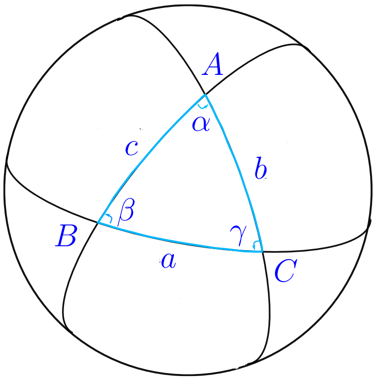
4.5 Without any computation, draw (approximately) an ellipse and its evolute. Explain your reasoning.

4.6 In this exercise, we will study a curve that has been the cornerstone of the design of highways and rail tracks, the so-called *clothoid curve* or *Euler's spiral.*

When driving at a constant speed on a highway (the car can be thought of as tracing out a curve in the plane), the position of the steering wheel determines the rate of "turning" of the velocity vector (and, hence, is related to the acceleration of the car's curve). The rate of turning of the wheel is then associated to the so called "jerk" of the curve (in a natural parametrization, this is the derivative in the curvature of the curve). When designing the portion of a highway that connects two given roads, one aims to minimize abrupt changes of the wheel, distributing the jerk as uniformly as possible along that portion. Thus, such a connecting curve has constant jerk; this curve is called the *clothoid curve.*

(a) Derive the differential equation satisfied by the Cartesian expression $\gamma(s) = (x(s), y(s))$ of the clothoid curve parametrized by unit speed.

(*b) Obtain an expression for $(x(s), y(s))$ with the initial conditions $\gamma(0) = (0, 0)$, $\dot{\gamma}(0) = (1, 0)$ and $\kappa_\gamma(0) = 0$ in terms of the special functions $\int \cos(s^2)\, ds$, $\int \sin(s^2)\, ds$ *(Hint: Instead of working with the differential equation in Cartesian coordinates, you might want to derive a differential equation for the angle $\theta(s)$ formed by the velocity vector and the x-axis and solve for that first.*



A *spherical triangle* is defined by three points A, B, C on a sphere $\mathbb{S}$, with the great-circle arcs a (joining B and C), b (joining A and C), and c (joining A and B). These arcs are the sides of the spherical triangle. We denote by α the angle formed at A by arcs b and c, similarly β at B and γ at C.

Recall: A great circle on a sphere is the circle obtained by intersecting the sphere with a plane passing through the center of the sphere. Other circles are called small circles. Two points on a sphere are always connected by two great-circle arcs; in defining a spherical triangle, we only consider the shorter one.

4.7 By abuse of notation, we also denote by a, b, and c the lengths of the sides of the spherical triangle. Prove the following spherical trigonometry formula:

$$\cos\left(\tfrac{c}{r}\right) = \cos\left(\tfrac{a}{r}\right)\cos\left(\tfrac{b}{r}\right) + \sin\left(\tfrac{a}{r}\right)\sin\left(\tfrac{b}{r}\right)\cos(\gamma),$$

where r is the radius of the sphere.

4.8 The spherical distance $d_{\mathbb{S}}(A, B)$ between two points A and B on a sphere $\mathbb{S}$ is by definition the length of the great-circle arc joining these points. Show, using spherical trigonometry, that $(\mathbb{S}, d_{\mathbb{S}})$ satisfies all the properties of a metric space.

Bonus exercise (not part of the examinable course material).

4.9 The goal of this exercise is to show that one can (re)define the length of a C^1 curve by a process of "polygonal approximations." Let $\gamma : [a, b] \to \mathbb{R}^n$ be a C^1 curve, and let $\sigma = [t_0 = a < t_1 < \cdots < t_m = b]$ be a subdivision of $[a, b]$. We set

$$L(\gamma) = \sup_{\sigma} \sum_{i=0}^{m-1} d(\gamma(t_i), \gamma(t_{i+1})),$$

where the supremum is taken over all subdivisions of $[a, b]$ and $d(p, q) = \|q - p\|$.

(a) Make a drawing and briefly explain the meaning of this formula.

(b) Show that for every C^1 curve we have $L(\gamma) \leqslant \ell(\gamma)$, where $\ell(\gamma)$ is the length of γ as defined in the course.

(c) Prove the reverse inequality $\ell(\gamma) \leqslant L(\gamma)$. (Hint: use that $\dot{\gamma}$ is uniformly continuous and show that for every $\varepsilon > 0$ one can find a sufficiently fine subdivision of $[a, b]$ such that

$$\ell(\gamma) \leqslant \sum_{i=1}^{m-1} d(\gamma(t_i), \gamma(t_{i+1})) + 2\varepsilon(b - a).$$

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